Counting cycles in planar graphs

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Euler's Formula

A graph drawn in the plane without crossing edges is a PLANE GRAPH.

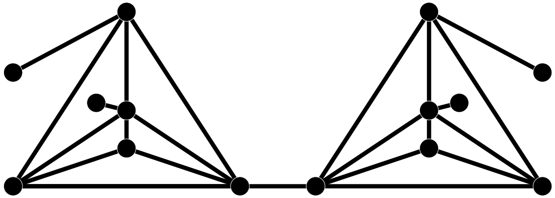
Let G be a plane graph with

- $V = V(G)$, the set of vertices,
- $E = E(G)$, the set of edges,
- $F = F(G)$, the set of faces.

Theorem (Leonhard Euler, 1758)

For any connected plane graph, $|V| - |E| + |F| = 2.$

A graph with a plane graph drawing is a PLANAR GRAPH.



Euler's Formula

Theorem (Leonhard Euler, 1758)

For any connected plane graph, $|V| - |E| + |F| = 2$.

Lemma (Handshaking Lemma for plane graphs)

For every plane graph G*,*

$$2|E| = \sum_{v \in V} \deg(v) = \sum_{f \in F} \deg(f).$$

Corollary

For every plane graph G *with* $|V| \geq 3$,

$$|E| \leq 3|V| - 6$$

$$|E| \leq 2|V| - 4, \qquad \textit{if } G \textit{ is bipartite.}$$

Euler's Formula

Corollary

For every plane graph G with $|V| \geq 3$,

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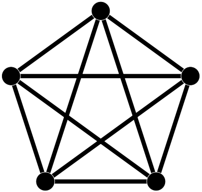
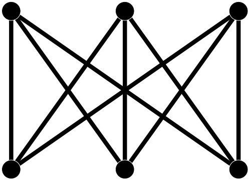
K_5



$|V| = 5$, $|E| = 10$



$K_{3,3}$

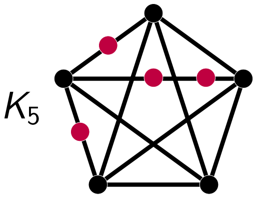
$|V| = 6$, $|E| = 9$

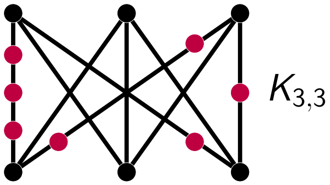
Euler's Formula

Theorem

If G is a graph, then G is planar iff

- *G has no* SUBDIVISION *of K_5 or $K_{3,3}$; [Kuratowski, 1930]*
- *G has no minor of K_5 or $K_{3,3}$. [Wagner, 1937]*





Hakimi-Schmeichel

Definition

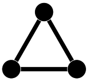
Let $\mathrm{N}_{\mathcal{P}}(n, H)$ *denote the maximum number of copies of* H *in an* n*-vertex* PLANAR *graph.*

Theorem

If $n \geq 3$*, then*

$$\mathrm{N}_{\mathcal{P}}(n, K_2) = 3n - 6.$$

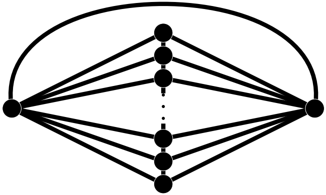
This is achieved by any planar triangulation.
Tutte proved there are $\frac{n^{-7/2}}{64\sqrt{6\pi}}\left(\frac{256}{27}\right)^{n-2}$ planar triangulations.



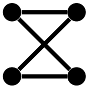
Theorem (Hakimi-Schmeichel, 1979)

If $n \geq 3$, then

$$\mathrm{N}_{\mathcal{P}}(n, C_3) = 3n - 8.$$



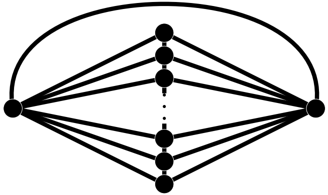
Hakimi-Schmeichel



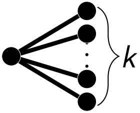
Theorem (Hakimi-Schmeichel, 1979)

If $n \geq 3$, *then*

$$\mathrm{N}_{\mathcal{P}}(n, C_4) = \mathrm{N}_{\mathcal{P}}(n, K_{2,2}) = \frac{n^2 + 3n - 22}{2}.$$



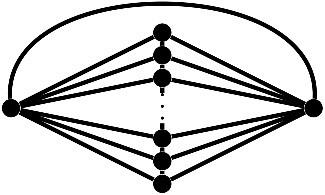
Alon-Caro



Theorem (Alon-Caro, 1984)

If $k \geq 3$ and $n \geq 4$, then

$$\mathrm{N}_{\mathcal{P}}(n, K_{1,k}) = 2 \cdot \binom{n-1}{k} + (n-4) \cdot \binom{4}{k} + 2 \cdot 3k.$$



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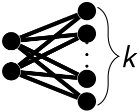
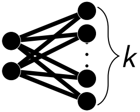


Theorem (Alon-Caro, 1984)

If $k \geq 3$ and $n \geq 4$, then

$$\mathrm{N}_{\mathcal{P}}(n, K_{2,k}) = \begin{cases} \binom{n-2}{k}, & \text{if } k \geq 5 \text{ or if } k = 4 \text{ and } n \neq 6; \\ 3, & \text{if } k = 4 \text{ and } n = 6; \\ \binom{n-2}{3} + 3(n-4), & \text{if } k = 3 \text{ and } n \neq 6; \\ 12, & \text{if } k = 3 \text{ and } n = 6; \\ \binom{n-2}{2} + 4n - 14, & \text{if } k = 2. \end{cases}$$

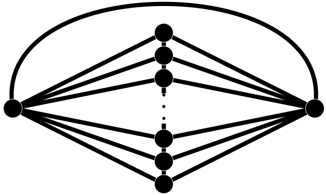
Alon-Caro



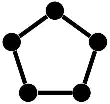
Theorem (Alon-Caro, 1984)

If $n \geq 4$, then

$$\mathrm{N}_{\mathcal{P}}(n, K_{2,k}) = \binom{n-2}{k} + O\left(n^{k-1}\right).$$

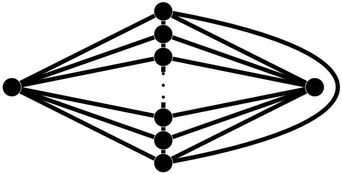


Five-cycles

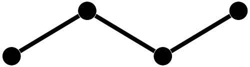


Theorem (Győri-Paulos-Salia-Tompkins-Zamora, 2019)

$$N_{\mathcal{P}}(n, C_5) = \begin{cases} 2n^2 - 10n + 12, & \text{if } n = 6 \text{ or } n \geq 8; \\ 41, & \text{if } n = 7; \\ 6, & \text{if } n = 5. \end{cases}$$



Four-paths



Theorem (Győri-Paulos-Salia-Tompkins-Zamora, 2019)

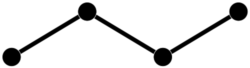
$$\mathrm{N}_{\mathcal{P}}(n,P_4)=\begin{cases}7n^2-32n+27, & \text{if } n\geq 9 \text{ or } n=5,6;\\ 222, & \text{if } n=8;\\ 147, & \text{if } n=7;\\ 12, & \text{if } n=4,\end{cases}$$

$$\mathrm{N}_{\mathcal{P}}(n,P_k)=\Theta\left(n^{\lfloor (k-1)/2\rfloor+1}\right).$$

Theorem (Huynh-Joret-Wood, 2022, generalized by C.-H. Liu, 2021+)

For all H, there exists a ℓ such that $\mathrm{N}_{\mathcal{P}}(n,H)=\Theta(n^{\ell})$.

Four-paths



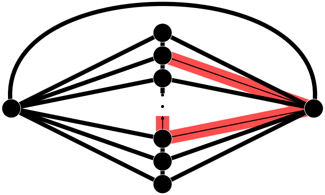
Theorem (Győri-Paulos-Salia-Tompkins-Zamora, 2019)

If $n \geq 9$, then

$$\mathrm{N}_{\mathcal{P}}(n, P_4) = 7n^2 - 32n + 27.$$

$7n^2 - 32n + 27$

$$= 2 \cdot 2 \cdot (n-3) \cdot (n-4) + 2 \cdot (n-2) \cdot (n-3) + 2 \cdot \binom{n-2}{2} + O(n)$$



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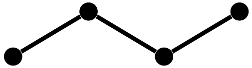

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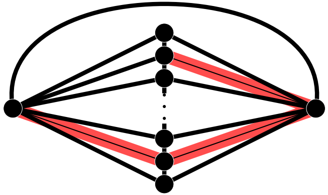

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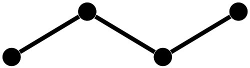

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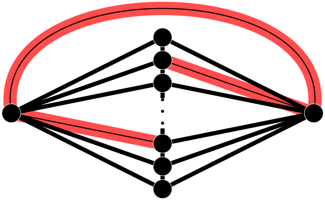
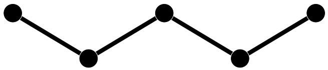
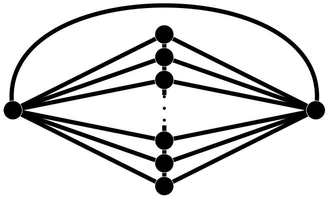


Five-paths



Theorem (Ghosh-Győri-M.-Paulos-Salia-Xiao-Zamora, 2021)

$$\mathrm{N}_{\mathcal{P}}(n, P_5) = n^3 + O\left(n^2\right)$$



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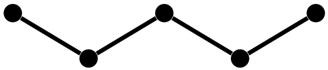

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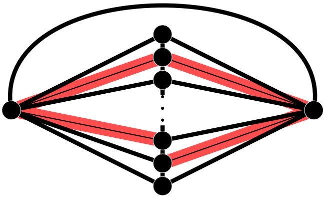
$$\mathrm{N}_{\mathcal{P}}(n, P_5) = n^3 + O\left(n^2\right)$$

$$n^3 + O\left(n^2\right) = (n-2)(n-3)(n-4) + O\left(n^2\right)$$



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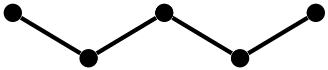

Theorem (Ghosh-Győri-M.-Paulos-Salia-Xiao-Zamora, 2021)

$$\mathrm{N}_{\mathcal{P}}(n, P_5) = n^3 + O\left(n^2\right)$$

Big idea of the proof:

Lemma

Let $n \geq k \geq 3$ and let G be a planar graph on n vertices such that $S \subseteq V(G)$, $|S| = k$. Then,

$$\sum_{v \in S} \deg(v) \leq 2n + 6k - 16.$$

Proof of five-paths

Theorem (Ghosh-Győri-M.-Paulos-Salia-Xiao-Zamora, 2021)

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Proof.

$$\sum_{v \in S} \deg(v) = \sum_{v \in S} \deg_{G[S]}(v) + e(S, V - S) \leq 2(3k-6) + (2n-4).$$

□

Proof of five-paths

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$$\sum_{v \in S} \deg(v) \leq 2n + 6k - 16.$$

If the vertices of G are $\deg(v_1) \geq \deg(v_2) \geq \cdots \geq \deg(v_n)$, then

$$\#P_5(G) \leq \sum_{i<j} \deg(v_i) \deg(v_i, v_j) \deg(v_j)$$

Proof of five-paths

Theorem (Ghosh-Győri-M.-Paulos-Salia-Xiao-Zamora, 2021)

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If the vertices of G are $\deg(v_1) \geq \deg(v_2) \geq \cdots \geq \deg(v_n)$, then

$$\begin{aligned}
\#P_5(G) &\leq \sum_{i<j} \deg(v_i) \deg(v_i, v_j) \deg(v_j) \\
&\leq \sum_{i<j} \deg(v_i) \min\{\deg(v_i), \deg(v_j)\} \deg(v_j) \\
&\leq \sum_{i<j} \deg(v_i) (\deg(v_j))^2
\end{aligned}$$

Proof of five-paths

Theorem (Ghosh-Győri-M.-Paulos-Salia-Xiao-Zamora, 2021)

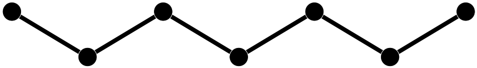
$$\mathrm{N}_{\mathcal{P}}(n, P_5) = n^3 + O\left(n^2\right)$$

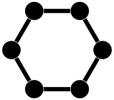
$$\begin{cases} \text{maximize} & \sum_{i<j} x_i x_j^2 \\ \text{subject to} & n \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0. \\ & \sum_{i=1}^{k} x_i \leq 2n + 6k - 16, \qquad \forall k \geq 3 \\ & \sum_{i=1}^{n} x_i \leq 6n - 12. \end{cases}$$

The solution is $n^3 + O\left(n^2\right)$.

(Our proof requires $n \geq 11664$.)

Seven-paths and six-cycles

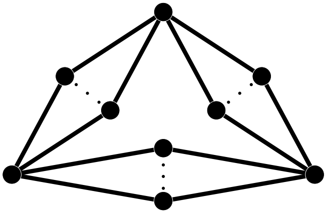




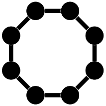
Theorem (Cox-M., 2022)

$$\mathrm{N}_{\mathcal{P}}(n, P_7) = \frac{4}{27} n^4 + O\left(n^{4-1/5}\right)$$

$$\mathrm{N}_{\mathcal{P}}(n, C_6) = \left(\frac{n}{3}\right)^3 + O\left(n^{3-1/5}\right)$$

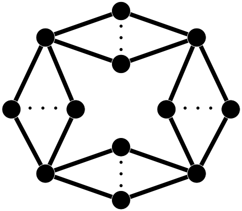


Eight-cycles



Theorem (Cox-M., 2022)

$$N_{\mathcal{P}}(n, C_8) = \left(\frac{n}{4}\right)^4 + O\left(n^{4-1/5}\right)$$

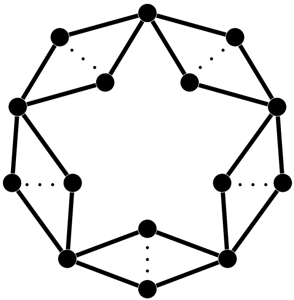


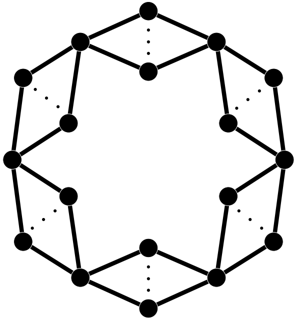
10- and 12-cycles

Theorem (Cox-M., 2023)

$$N_{\mathcal{P}}(n, C_{10}) = \left(\frac{n}{5}\right)^5 + o\left(n^5\right)$$

$$N_{\mathcal{P}}(n, C_{12}) = \left(\frac{n}{6}\right)^6 + o\left(n^6\right).$$

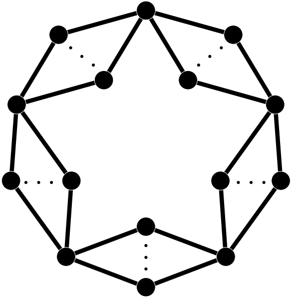


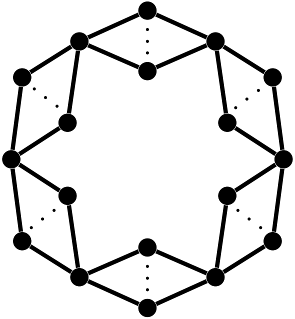


10- and 12-cycles

Theorem (Lv-Győri-He-Salia-Tompkins-Zhu, 2024)

$$\mathrm{N}_{\mathcal{P}}(n, C_{2k}) = \left(\frac{n}{k}\right)^k + o\left(n^k\right).$$

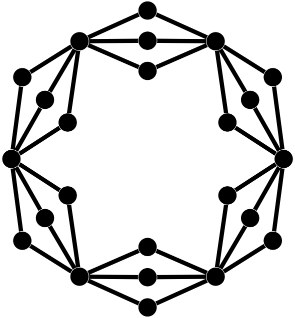




Other results

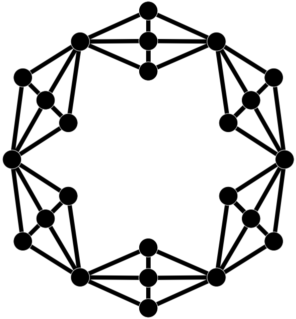
Definition

Let H be a planar graph. $H\{k\}$ denotes the "planar blow-up of H by k". That is, replace each edge of H with a $K_{2,k}$.



$C_6\{3\}$

Other results



Objective

A "reduction lemma" that says $\mathrm{N}_{\mathcal{P}}(n, H)$ *is asymptotically maximized by*

- *taking some graph* G,
- *blowing up the edges by different amounts, and*
- *putting a path inside each blowup.*

Probability mass on a graph

Definition

- *Let V be vertex set and let $\mu : \binom{V}{2} \to [0,1]$ be a probability mass. That is, $\sum \left\{ \mu(e) : e \in \binom{V}{2} \right\} = 1$.*

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$$\mu(G) = \prod_{e \in E} \mu(e).$$

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- *For a vertex $x \in V$, let*

$$\bar{\mu}(x) = \sum_{y \in V - \{x\}} \mu(xy).$$

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Theorem (Even Cycle Reduction Lemma)

Let $m \geq 3$.

- $$\beta(\mu; m) \stackrel{\text{def}}{=} \sum_{C \in \mathrm{C}(K_V)} \prod_{e \in E(C)} \mu(e) = \mathbb{P}(m \text{ edges form } C_m)$$
- $$\beta(m) \stackrel{\text{def}}{=} \sup \left\{ \beta(\mu; m) : \operatorname{supp} \mu \subset \binom{V}{2} \text{ for a finite } V \right\}$$
- $$\mathrm{N}_{\mathcal{P}}(n, C_{2m}) \leq \beta(m) \cdot n^m + O\left(n^{m-1/5}\right)$$

Probability mass on a graph

Definition

- *Let V be vertex set and let $\mu : \binom{V}{2} \to [0,1]$ be a probability mass.*
- *For a graph $G = (V, E)$, define $\mu(G) = \prod_{e \in E} \mu(e)$.*
- *For a vertex $x \in V$, let $\bar{\mu}(x) = \sum_{y \in V - \{x\}} \mu(xy)$.*

Theorem (Odd Path Reduction Lemma)

Let $m \geq 2$. Then $\mathrm{N}_{\mathcal{P}}(n, P_{2m+1}) \leq \rho(m) \cdot n^{m+1} + O\left(n^{m+4/5}\right)$ where

$$\rho(\mu; m) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{\mathrm{v} \in (V)_m} \bar{\mu}(x_1) \left(\prod_{i=1}^{m-1} \mu\left(x_i x_{i+1}\right) \right) \bar{\mu}(x_m)$$

Notes about reduction lemmas

- Reduction lemmas are not structural, but counting lemmas.
- We can't guarantee that, e.g., μ isn't positive on a K_5, non-planar.
- Karush-Kuhn-Tucker (Lagrange multipliers) used to compute ρ or β.

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Examples for the cycle C_{2m}

- If μ is an equal distribution on $E(K_m)$, then

$$\sum_{C \in \mathrm{C}(K_m)} \prod_{e \in E(C)} \mu(e) = \frac{(m-1)!}{2} \left(\frac{1}{\binom{m}{2}} \right)^m .$$

- If μ is an equal distribution on $E(C_m)$, then

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The largest is $\left(\frac{1}{m}\right)^m$ for $m \geq 3$.

- $m = 3, 4, 5, 6$ [Cox-M.]
- $m \geq 7$ [Lv-Győri-He-Salia-Tompkins-Zhu].

Odd cycle Reduction Lemma

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Theorem (Odd Cycle Reduction Lemma: Heath-M-Wells, 2023+)

Let $m \geq 2$.

- $$\begin{aligned} \beta(\mu; m) &\stackrel{\text{def}}{=} 2m \cdot \sum_{C \in \mathrm{C}(K_V)} \prod_{e \in E(C)} \mu(e) + \sum_{P \in \mathrm{P}(K_V)} \prod_{e \in E(P)} \mu(e) \\ &= 2m \cdot \mathbb{P}(m \text{ edges form } C_m) + \mathbb{P}(m \text{ edges form } P_{m+1}) \end{aligned}$$
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Theorem (Heath-M.-Wells, 2023+)

$$\beta(m) = \frac{2}{m^{m-1}}, \qquad \text{if } m \in \{3, 4\};$$
$$\beta(m) \leq \frac{2.7}{m^{m-1}}, \qquad \text{if } m \geq 5.$$

Odd cycle Reduction Lemma

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Odd cycle Reduction Lemma

Theorem (Heath-M.-Wells, 2023+)

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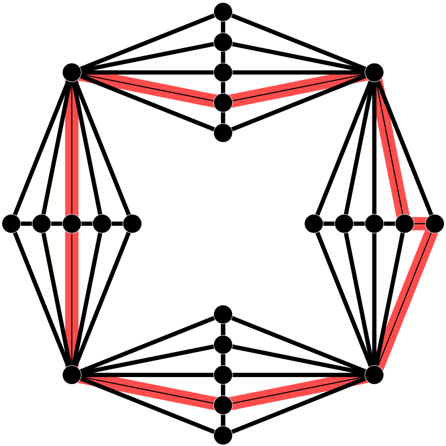

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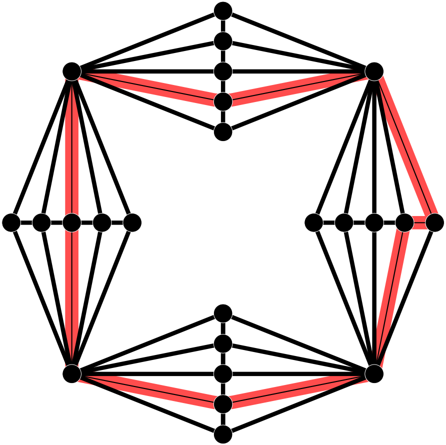

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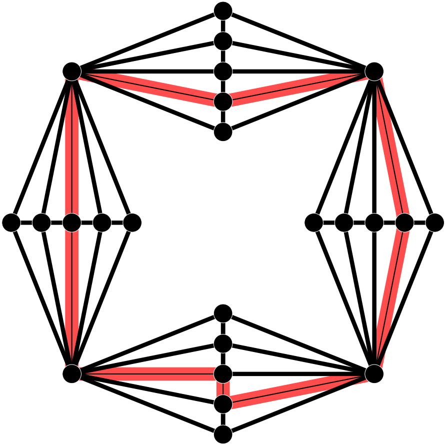

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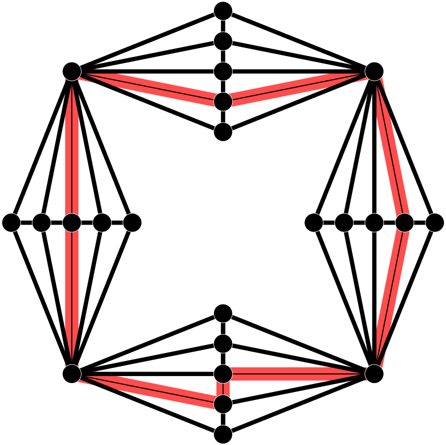

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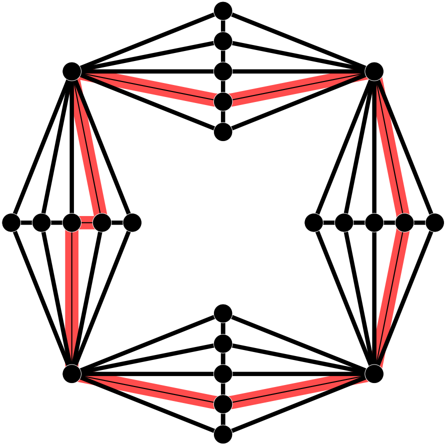

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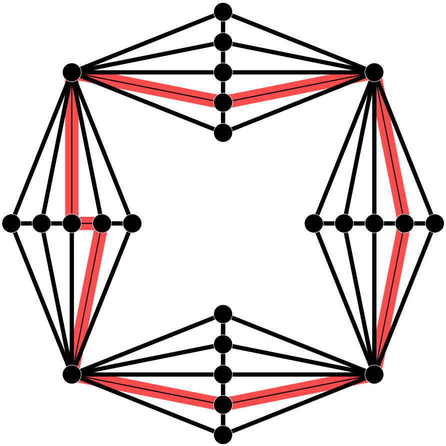

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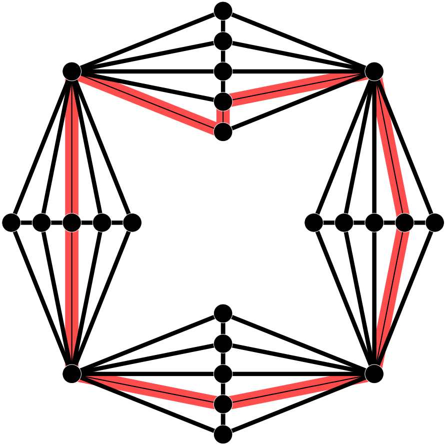

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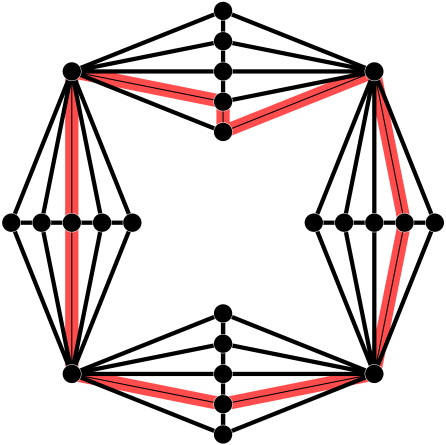

Open Problems

Question

- *What is* $\mathrm{N}_{\mathcal{P}}(n, P_m)$ *for* $m = 9, 11, \ldots$*?* *[Reduction lemma exists]*
- *What is* $\mathrm{N}_{\mathcal{P}}(n, C_m)$ *for* $m = 11, 13, \ldots$*?* *[Reduction lemma exists]*
- *What is* $\mathrm{N}_{\mathcal{P}}(n, P_m)$ *for* $m = 6, 8, \ldots$*?* *[No reduction lemma yet]*

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Question (General maximum likelihood)

Given a graph H *on* m *edges.*

- *For a probability distribution* μ *on* $\binom{V}{2}$*, choose* m *edges and compute* $\mathbb{P}(\text{a copy of } H) = \beta(\mu, H)$*.*
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This is the MAXIMUM LIKELIHOOD *question.*

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Although a μ achieving the maximum often occurs, it doesn't have to.

Observation

- *If $H = K_{1,m}$ then the maximum likelihood results from a sequence $\{\sigma_n\}$ of stars with each edge having probability $1/n$.*
- *If $H = m \times K_2$ then the maximum likelihood results from a sequence $\{\mu_n\}$ of matchings with each edge having probability $1/n$.*

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Remark

- *Finding $\rho(\mu; m)$, which counts odd paths is not exactly this kind of maximum likelihood question.*
- *The odd cycle C_{2m+1} problem requires computing $2m \cdot \beta(\mu, C_m) + \beta(\mu, P_{m+1})$.*

Thank you!